Algebra-I
B. Math - First year
Mid-Semestral Exam

Time: 3hrs
Max score: 100

Answer question **1** and any **three** from the rest.

(1) Decide whether the following statements are TRUE or FALSE. Justify. No points will be awarded in the absence of a correct justification.
(a) There exists a group that has exactly 5 elements of order 5.
(b) A group of order 30 has at most 7 subgroups of order 5.
(c) S_4 contains a normal subgroup of order 3.
(d) If p is the smallest prime dividing the order of G then any subgroup of index p is normal.
(e) $\mathbb{Q}$ contains a subgroup of finite index. 5+5+5+5+5

(2) (a) Let $S^1 \subset \mathbb{C}^\times$ be the circle group. Show that
(i) $\mathbb{R}/\mathbb{Z} \cong S^1$.
(ii) There is a subgroup of S^1 which is isomorphic to $\mathbb{Q}/\mathbb{Z}$. Describe the elements of this subgroup.
(b) Show that if G is a group with A, B subgroups of G, such that $A \subseteq N_G(B)$, then $A \cap B$ is a normal subgroup of A and $AB/B \cong A/A \cap B$.
(c) Let G be a group, let N be a normal subgroup of G, and let $K \subseteq G$ be another subgroup. Let $\pi : G \longrightarrow G/N$ be the canonical homomorphism. Show that $\pi(K) = KN/N$. 10+8+7

(3) (a) Show that two elements of S_n are conjugate in S_n if and only if they have the same cycle type.
(b) Hence deduce that the number of conjugacy classes of S_n equals the number of partitions of n.
(c) Compute $o(C_{S_n}((12)(34)))$, $n \geq 4$, where $C_G(a)$ denotes the centralizer of the element a in G. 10+5+10

(4) Let G be a finite group operating on a finite set X. For each element $g \in G$, let X^g denote the subset of elements of X fixed by g that is, $X^g = \{x \in X/\ gx = x\}$.
(a) Prove that $\sum_{x \in X} o(G_x) = \sum_{g \in G} |X^g|$.
(b) Prove that $o(G)$.(number of orbits) $= \sum_{g \in G} |X^g|$. 10+15

(5) Let H be a normal subgroup of a group G.
(a) Show that G acts on H by conjugation i.e., for $g \in G, h \in H$, $(g, h) \longmapsto ghg^{-1}$ defines an action of G on H.

(b) Hence show that $G/C_G(H)$ is isomorphic to a subgroup of $Aut(H)$.
(c) Deduce that if $o(H) = 5$ and if $o(G)$ is odd, then $H \subseteq Z(G)$. 5+10+10